Georg Fantner

Week 6: Applying strain and stress in multiple dimensions

1. Properties of materials
2. Limiting behavior
3. Torsion

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Limiting behavior and failure

Properties of materials

Stiffness: the property that enables a material to withstand high stress without great strain. It results in a steep first part of the stress strain curve. The stiffness is a function of the elastic modulus

Strength: determines the greatest stress that a material can withstand before failure. It depends on the material and the situation if failure refers to the yield point or the fracture point.

Elasticity: enables a material to regain its original dimensions after a deforming load is removed

Brittleness: the absence of any plastic deformation before abrupt failure

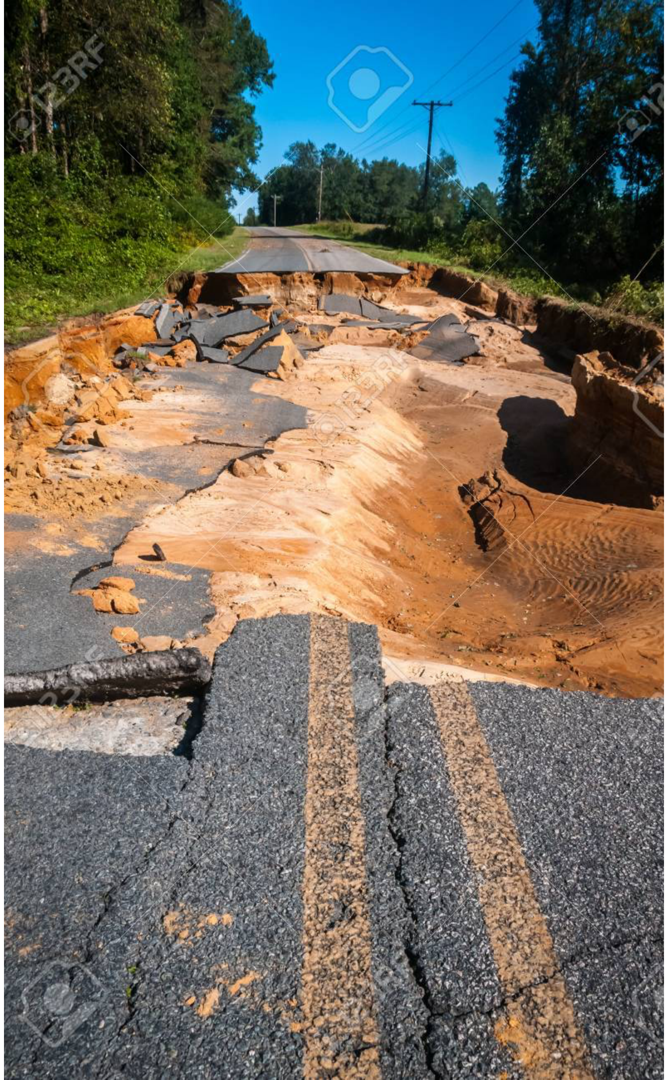
Properties of materials

Ductility: describes the amount of *plastic* deformation that a material can undergo in *tension* before rupture

Malleability: describes the amount of *plastic* deformation that a material can undergo in *compression* before rupture

Toughness: the amount of energy that is required to crack a material. It enables a material to withstand high impact loads. In such loads, some of the impact energy is transferred and absorbed by the body.

Resilience: the ability of a material to endure high impact loads without inducing a stress above the elastic limit. In resilient materials, the energy of the impact is stored in the body and recovered when the body is unloaded.

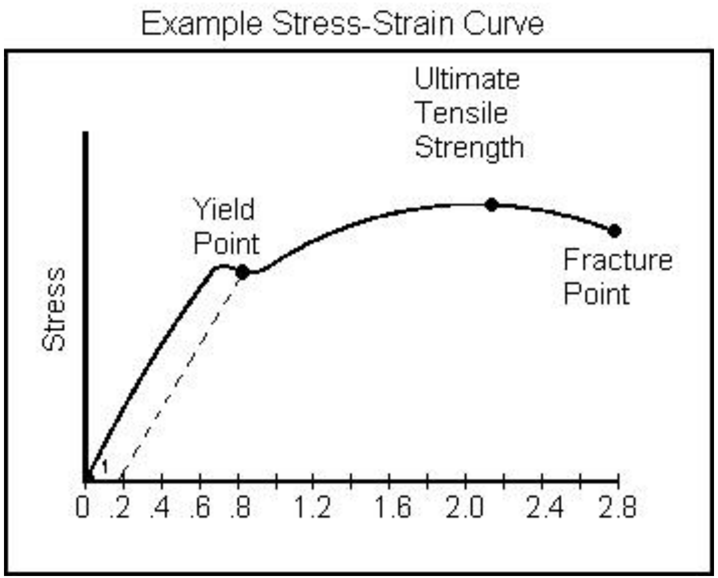


Limiting behavior - What is failure?

The stress or strain at which a structure fails depends very much on the application!

- Failure is generally defined as no longer fulfilling the desired function.
- Deviation from the desired function of a structure is defined with respect to *failure criteria:*
  - *what is the maximum allowable strain for a given load?*
  - *what is the maximum allowable stress?*
  - …
- Here we are concerned with the *materials aspect* of the failure

The stress/strain curve

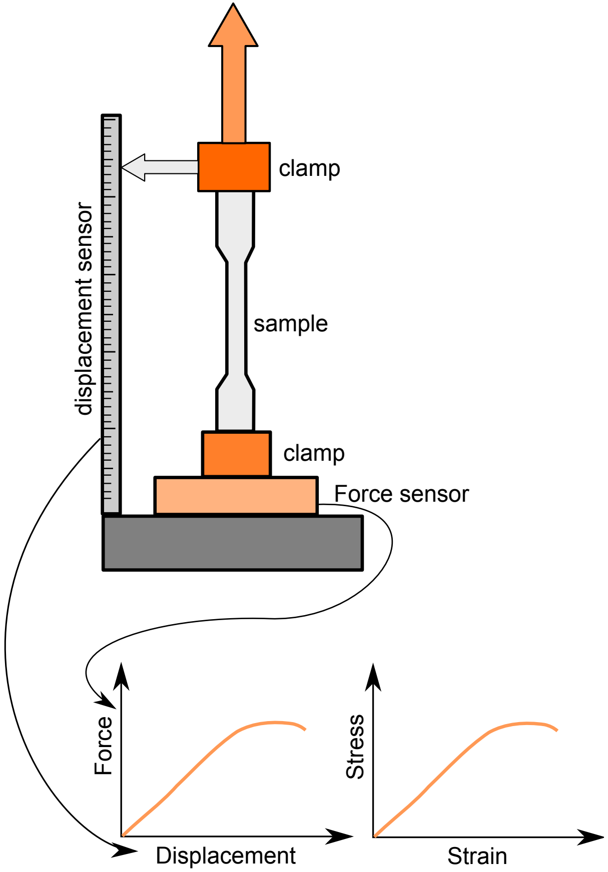


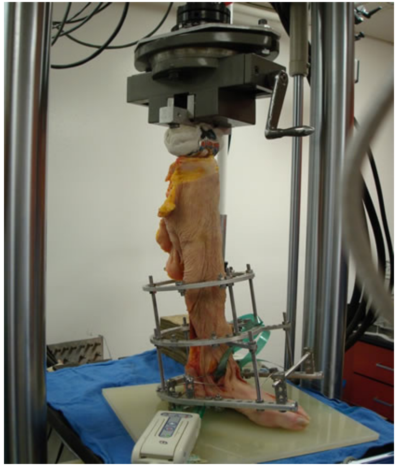
- Hooke's law only describes the early region of the stress/strain curve
- Beyond the yield point much less stress is required to obtain a large strain
- fracture point ≠ yield point
- "failure point" is a matter of definition for different applications
- Stress strain curves are recorded through experimental mechanical testing

Stress strain curves are determined through mechanical testing

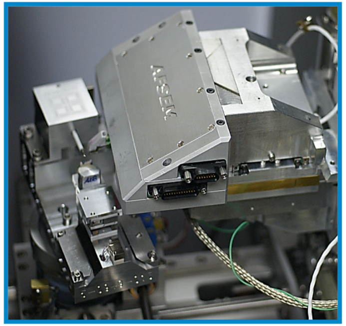
General purpose tensile and compression tester

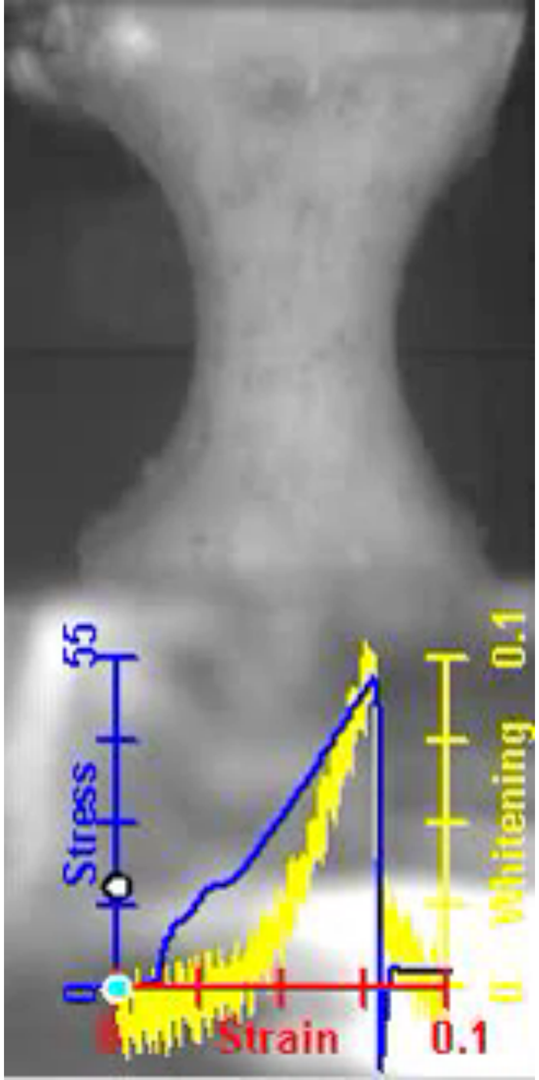
In vivo testing





Nanomechanical testing

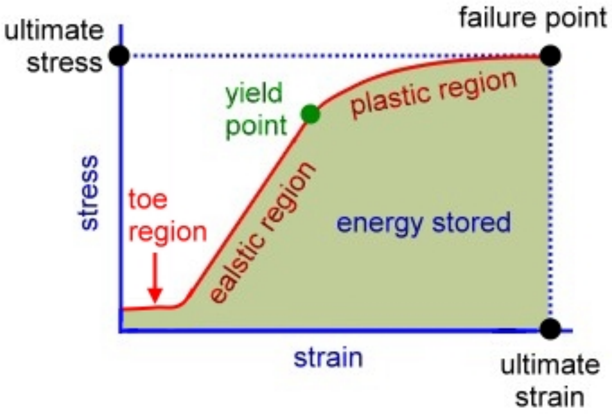




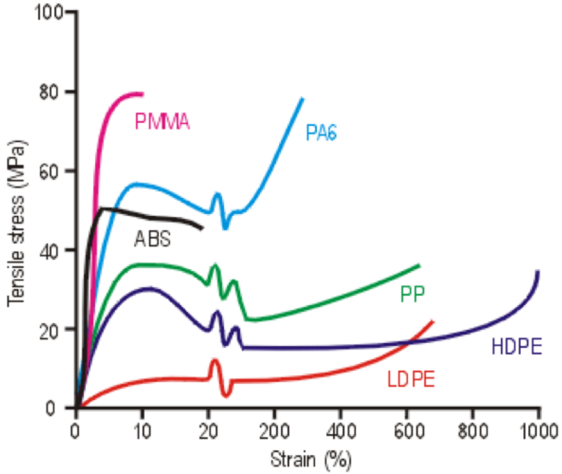
Tensile test of demineralized bone

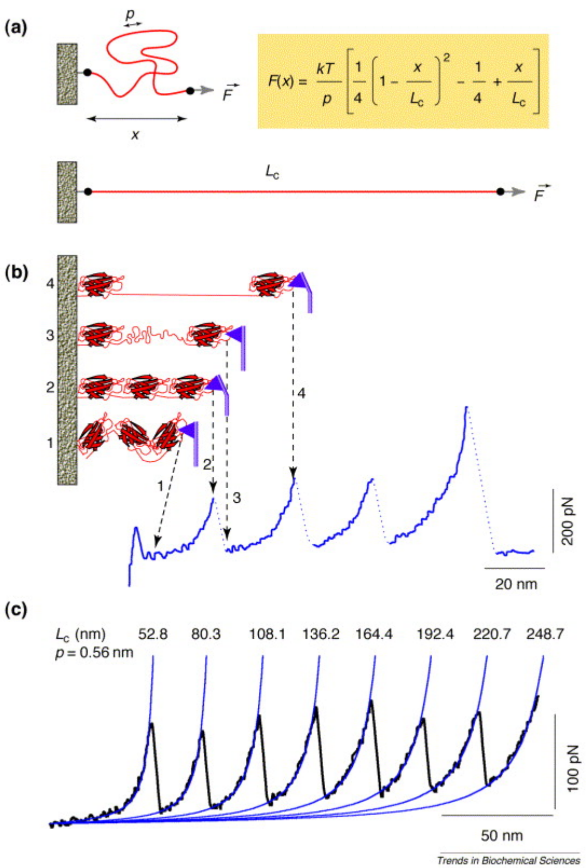
Stress-strain tests are often accompanied by some form of imaging method to observe local and global deformations.

Complex stress/strain curves



Stress-Strain Curve of Collagen Fiber

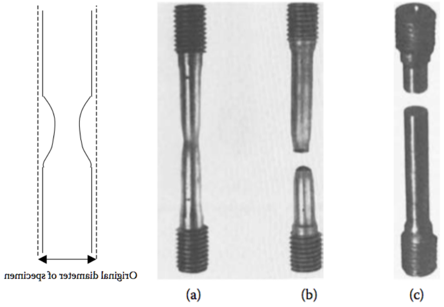


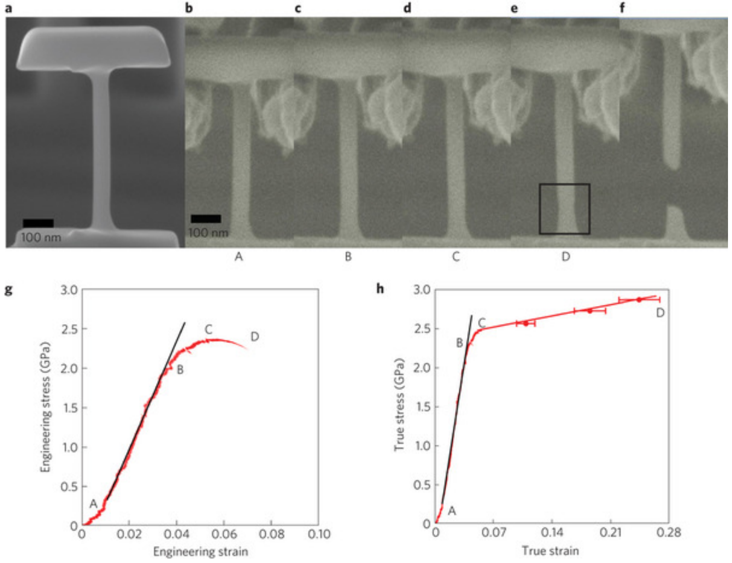


Trends in biochemical sciences
Volume 24, Issue 10, 1 October 1999, Pages 379–384

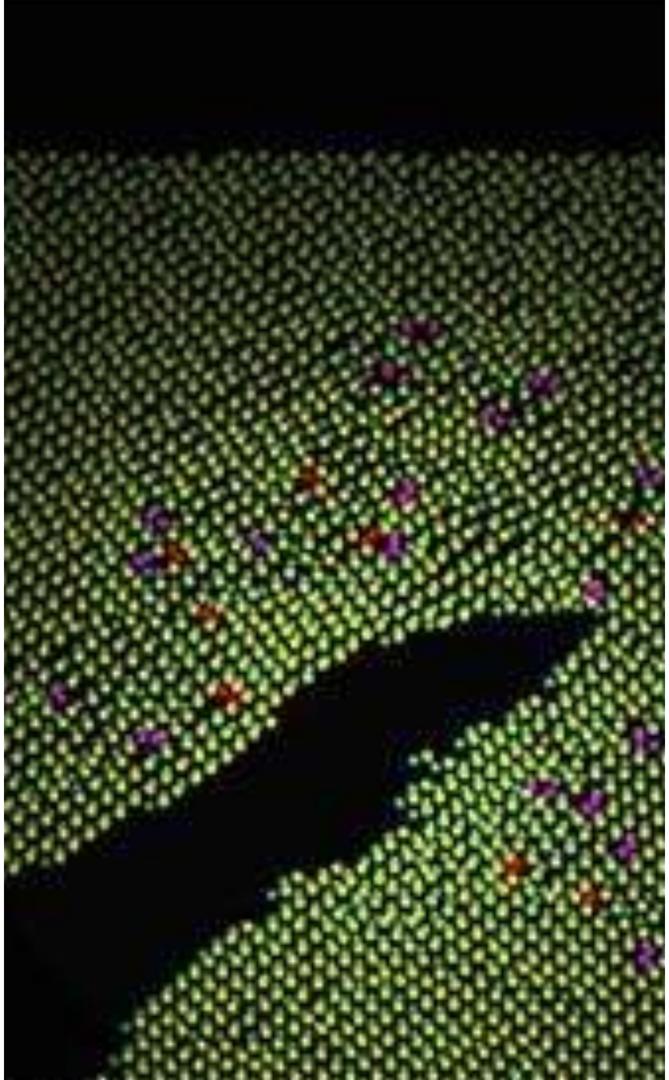
Macroscopic manifestations of stress/strain curve

- "Necking": before failure of ductile materials, they exhibit necking, which is a result of the Poisson effect in combination with stress concentration effect





Jang et al. *Nature Materials* ***9, 215–219 (2010)***
Transition from a strong-yet-brittle to a stronger-and-ductile state by size reduction of metallic glasses



Atomic origin of fracture strength

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- The fracture strength of a brittle elastic material ideally depends on the intermolecular bond strength between the molecules
- From theory: the fracture strength should be 1/10 of the elastic modulus E
- In reality: the fracture strength is 1/100 to 1/10.000 of E
- The fracture strength is strongly determined by flaws in the material (crystal dislocations, microcracks, impurities) that lead to *stress concentration*

Limiting behavior

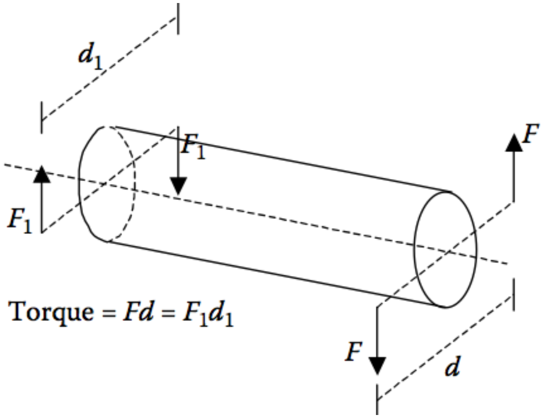
- Real world loading conditions are different than the loading conditions in which the materials values are measured in the laboratory
- We need reliable criteria that predict when a structure fails and when not.
- Different geometries and materials require different failure criteria
- Maximum normal stress criterion: failure is predicted when the maximum of the 3 normal (principal) stresses reaches the materials ultimate tensile or compressive strength
- Safety factor: describes the structural capacity of a system beyond the expected loads

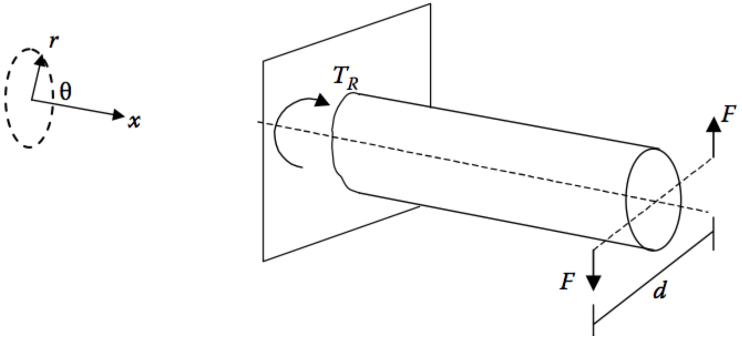
$$\text{safety factor (SF)} = \frac{\text{failure or yield stress}}{\text{maximum allowable or expected stress}}$$

- For a robust design, the safety factor should be > 2

Applying strain and stress in multiple dimensions

1. Torsion
2. Strain in torsion of a round bar
3. Stress in torsion of a round bar
4. Stress in a thin walled tube



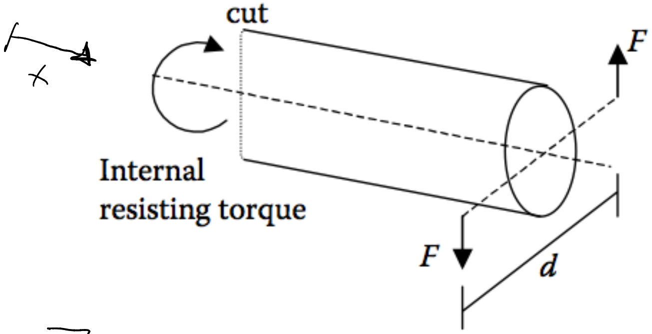


Torsion

When a structure is loaded by a pair of *couples* or a single couple and a fixed support we call the structure in torsion.

Couple: a pair of equal and parallel forces that act in opposite direction and tend to produce a rotation

Torsion



$$\sum M = 0$$

$$\sum M_x = 0$$

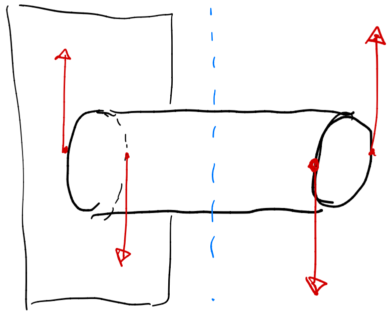
The equation of equilibrium in this case is (with x being the axis along the rotation axis of the member in question) :

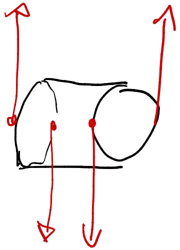
$$\sum M_x = 0$$

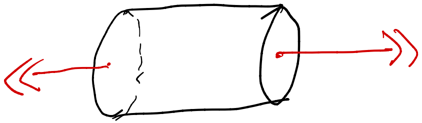
Using the methods of sections we see that an internal torque must exist that balances the external torque. This torque must be equal and opposite to the external torque.

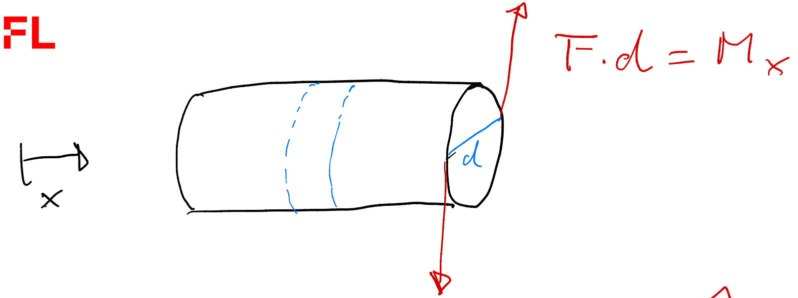
Torsional shear stress

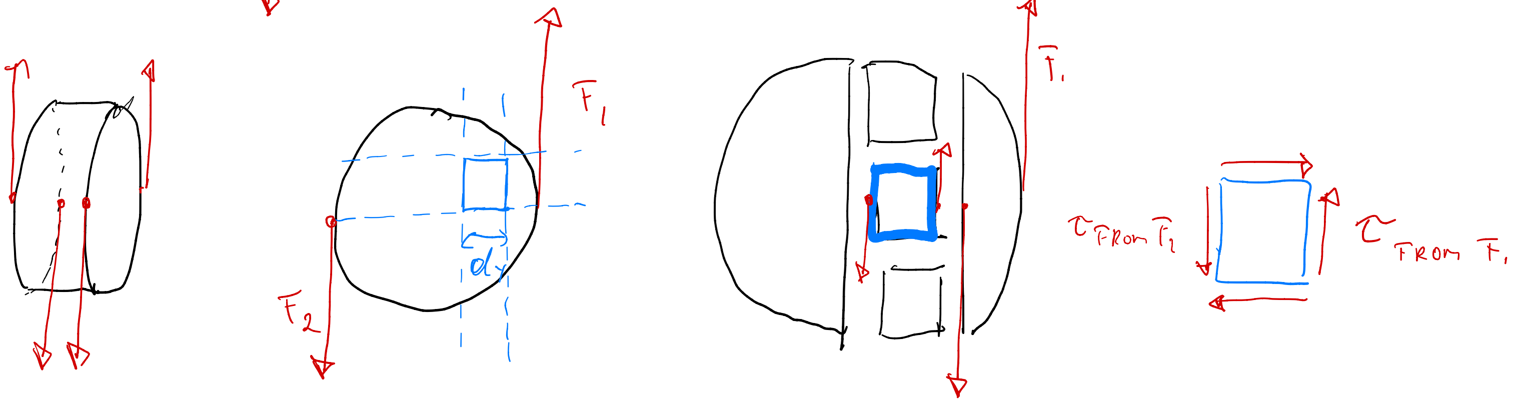
- Torsion causes no direct tension or compression in the material: it creates *pure shear stresses* on each crossectional plane
- If we apply the method of sections to an object to “cut it into slices”, and apply a torque, then the internal resistance that keeps the stack of slices from rotating is the *torsional shear stress*.
- The result of the torsional shear stress on any (internal) crossectional plane is an internal resisting torque.











THE EFFECT OF TORSIONAL LOADING ARE ONLY SHEAR STRESSES.

⇒ THESE SHEAR STRESSES CAN INDUCE SHEAR STRAINS

Useful statements about torsion

$$\tau = G \cdot \gamma$$

- A plane section perpendicular to the body/torsion axis remains plane after torque is applied
- The shear strain $\gamma(r)$ varies linearly from 0 at r=0 to γ_{max} on the outer edge
- For a linearly elastic material we can apply Hooke's law, where τ is the shear stress, γ is the shear stain and *G* the modulus of rigidity:

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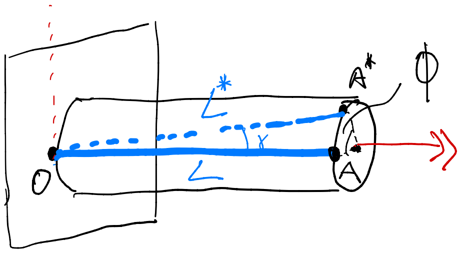
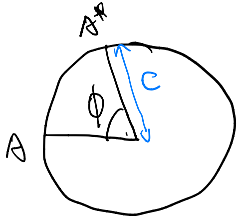
① RELATE TORSIONAL SHEAR STRAIN TO A MEASURABLE QUANTITY : ANGLE OF ROTATION

- LET'S ASSUME A CYLINDER FIXED ON ONE END WITH RADIUS $r = c$

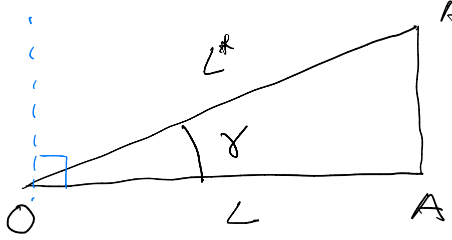


FRONT FACE :



$AA^* = \text{ARCLENGTH} = \phi c$

UNROLLED SIDE VIEW :



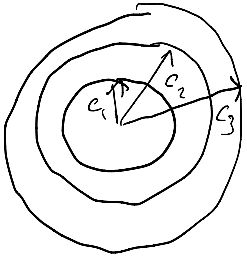
$AA^* = L \cdot \tan \gamma$

$$\phi c = L \cdot \underbrace{\tan \gamma}_{\approx \gamma}$$

$$\boxed{\gamma_{MAX} = \frac{\phi c}{L}}$$

SHEAR STRAIN ON THE OUTSIDE EDGE OF THE CYLINDER

$\gamma(r)$



$$\gamma_{MAX,i} = \frac{\phi c_i}{L}$$

$$\lim_{c_i \to 0} \gamma_{MAX,i} = \lim_{c \to 0} \frac{\phi c}{L} = 0$$

$\Rightarrow$ SHEAR STRAIN IS 0 AT $r=0$

$\Rightarrow$ SHEAR STRAIN IS LIN. PROP. TO C

$$\boxed{\gamma(r) = \frac{r}{c}\gamma_{MAX} = \frac{\phi r}{L}}$$

② APPLY HOOKE'S LAW IN TORSION:

RELATES TORSIONAL STRAIN TO TORSIONAL STRESS

$$\tau = G \cdot \gamma$$

$$\tau(r) = G\,\gamma(r) = G\,\frac{\phi r}{L} = \frac{r}{c}\,\tau_{MAX}$$

⇒ MAXIMUM SHEAR STRESS IS ALSO ON OUTSIDE OF CYLINDER

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③ RELATE THE TORSIONAL LOAD (TORQUE) TO THE TORSIONAL SHEAR STRESS

- STRESS = INTERNAL RESISTANCE TO THE EXTERNALLY APPLIED LOADS

=) THE SUM OF ALL STRESSES MULTIPLIED BY THEIR AREA EQUALS THE TOTAL INTERNAL FORCE

$$\int_A \tau(r) \cdot dA = F$$ ... TOTAL INTERNAL FORCE

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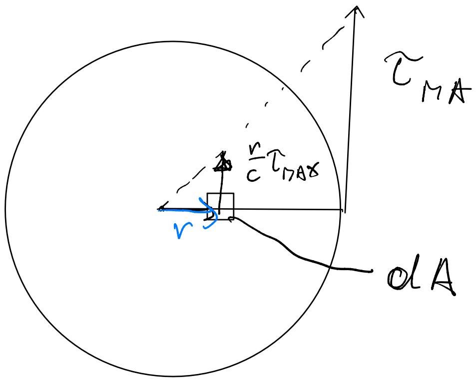
FOR TORSIONAL EQUILIBRIUM WE NEED TO BALANCE ALL THE MOMENTS: $\sum M = 0$



$$\vec{M} = \vec{F} \times \vec{r} = F \cdot r \cdot \sin\alpha$$

TORQUE $T = |\vec{M}|$

$$M = \int_A \underbrace{\tau(r)\, dA}_{\text{FORCE}} \cdot \underbrace{r}_{\text{FORCE ARM}} \cdot \underbrace{\sin\alpha}_{=1}$$

$$M = \int_A \frac{r}{c}\tau_{MAX}\, dA \cdot r = T$$

$c = \text{CONST}$

$\tau_{MAX} = \text{CONST}$

$$T = \frac{\tau_{max}}{c} \int_A r^2\, dA$$

$\int_A r^2\, dA$ → := SECOND MOMENT OF AREA J

$$T = \frac{\tau_{MAX}}{c} J$$

$$\Rightarrow \boxed{\tau_{MAX} = \frac{T c}{J}}$$

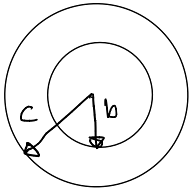
- Circular cross-section:

$$J = \int_A r^2 dA = \int_0^c 2\pi r^3 dr = 2\pi \frac{r^4}{4}\Big|_0^c = \frac{\pi}{2} c^4 = \frac{\pi d^4}{32}$$

Torsion formula for a circular shaft:

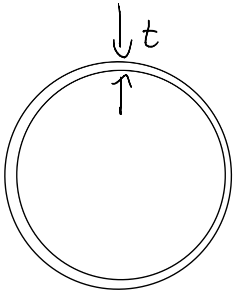
$$\boxed{\tau(r) = \frac{T r}{J} = 32 \frac{T r}{\pi d^4}}$$

J FOR A HOLLOW TUBE:



$$J = \int_A r^2 \, dA = \int_b^c 2\pi r^3 \, dr = 2\pi \frac{r^4}{4} \Big|_b^c = \frac{\pi}{2}\left(c^4 - b^4\right)$$

$$J_{TUBE} = J_{BAR\ r=c} - J_{BAR\ r=b}$$



THIN WALLED TUBE WITH THICKNESS t

$b \approx c \qquad c - b = t \qquad t < 0.1 \cdot c$

$$J \approx 2\pi R_{AV}^3 \, t$$

$$R_{AV} = \frac{b + c}{2}$$

④ Torsion Formula: Relate torque T to angle of twist ϕ

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We know:

1. $\gamma_{max} = \frac{c\phi}{L}$

2. $\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$

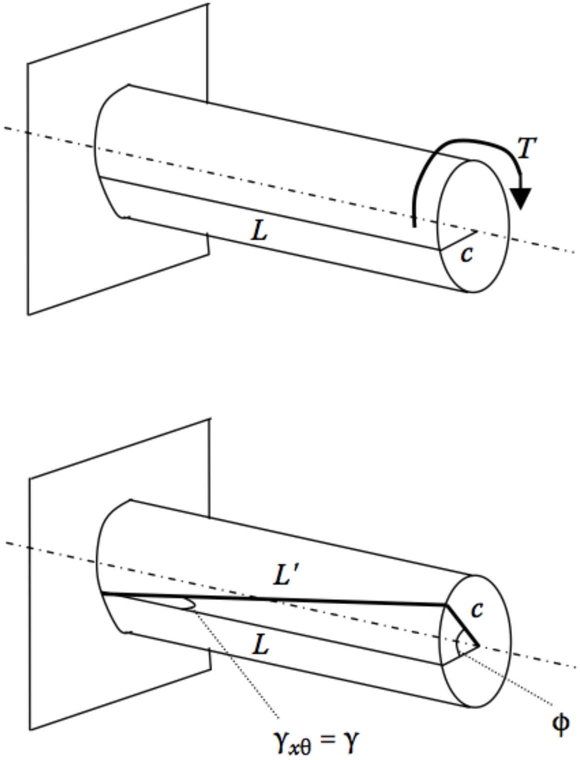
$$\frac{c\phi}{L} = \frac{Tc}{JG}$$

$$\phi = \frac{T \cdot L}{JG}$$

Hooke's law for a cylindrical bar in torsion

- For varying cross-sections or materials

$$\phi = \int_0^L \frac{T}{J(x) \cdot G(x)} \, dx$$



Torsional shear strain Relating torsion angle to torsional shear strain

Before we can derive a formula for the shear strain, we first have to define the geometric shear strain. We do this for a solid cylinder fixed at one end.

The shear strain on the outside of the cylinder is then the change of the (initially right) angle between the line L and the vertical.

$$\gamma_{max} = \frac{\phi c}{L} \qquad \gamma(r) = \frac{\phi r}{L}$$

Φ and γ are in radians

Relating torsional shear strain to torsional shear stress

- We have assumed:
  - γ varies linearly with r
  - A straight line on a plane that is parallel to the front plane will remain a straight line

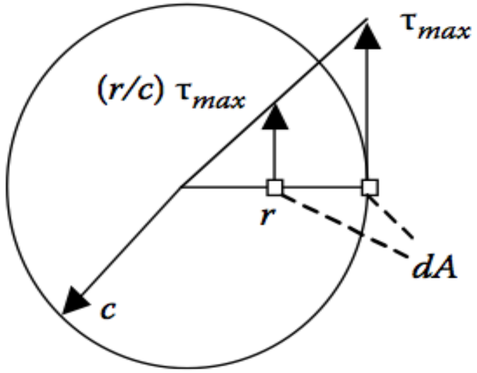
$$\gamma(r) = \frac{r}{c}\gamma_{max}$$

- We can therefore write with Hooke's law:

$$\tau(r) = G\frac{r}{c}\gamma_{max} = \frac{r}{c}\tau_{max}$$

The torsion formula: relating torque to torsional shear stress

- We can visualize the shear stress distribution on the circular cross-section:
- REMEMBER: stress is the internal resistance to applied loads (per unit area)
- Assume an infinitesimally small area dA, the the force actiing on this area dA is



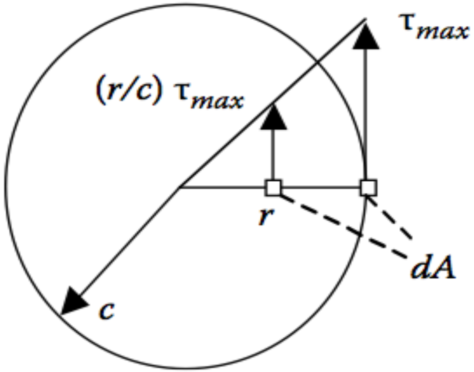
$$F_{dA} = \tau(r)dA = \frac{r}{c}\tau_{max}dA$$

- The equilibrium condition must be satisfied:

$$\sum M_x = 0 = \sum \vec{r} \times \vec{F}$$

-

The torsion formula: relating torque to torsional shear stress



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- The equilibrium condition must be satisfied:

$$\sum M_x = 0 = \sum \vec{r} \times \vec{F}$$

The torsion formula: relating torque to torsional shear stress

- The internal moment has to balance the externally applied torque T:

$$\begin{aligned} T &= \vec{F} \times \vec{r} = F r \sin\alpha \\ &= \underbrace{\int_A \frac{r}{c} \tau_{max} dA}_{Force} \cdot \underbrace{r}_{\text{moment arm}} \cdot \underbrace{\sin\alpha}_{\alpha=90^o} \\ &= \frac{\tau_{max}}{c} \underbrace{\int_A r^2 dA}_{J} \end{aligned}$$

- J is the second moment of inertia (polar moment of inertia)

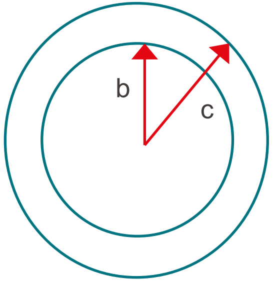
- The torsion formula is then:

$$\tau_{max} = \frac{Tc}{J}$$

$$\tau(r) = \frac{Tr}{J}$$

The torsion formula for a thin walled tube

- For a thin walled tube, the torsion formula can be approximated by:



$$
\begin{aligned}
\tau &= \frac{Tr}{J} \approx \frac{T}{2\pi r^2 t} \\
J &\approx 2\pi r^3 t \\
&\text{with} \\
t &= (c-b) \text{ and} \\
r &= \frac{b+c}{2}
\end{aligned}
$$

Hooke's law for torsion

- From Hooke's law we know:
- From the torsion formula we know:
- From the strain-assumption we know:

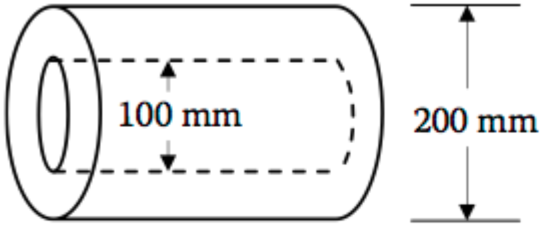
$$\gamma_{max} = \frac{\tau_{max}}{G}$$

$$\tau_{max} = \frac{Tc}{J}$$

$$\gamma_{max} = \frac{c\phi}{L}$$

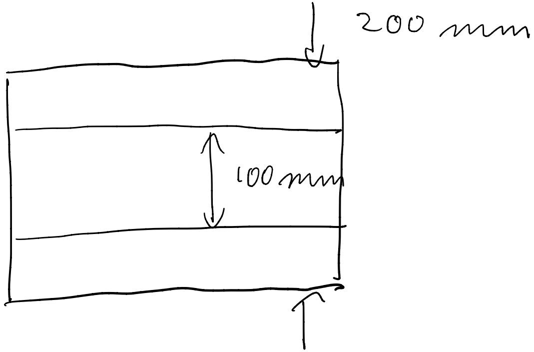
- We then get Hooke's law in torsion for a cylindrical bar:

$$\boxed{\phi = \frac{TL}{GJ}}$$



Examples for Torsion: Hollow shaft

A 100-mm-diameter core is bored out from a 200-mm-diameter solid circular shaft (Figure 4.35). What percentage of the shaft's torsional strength is lost due to this operation?



GIVEN: GEOMETRY 1: 200mm SOLID BAR

GEOMETRY 2: 200 mm OUTER DIAMETER, 100mm INNER DIAM.

ASKED: DIFFERENCE IN STRENGTH OF THE TWO STRUCTURES

GOV. PRINC: TORSION FORMULA: $\tau_{MAX} = \frac{Tc}{J}$

ANSWER:

$$\tau^{(1)}_{MAX} = \frac{T^{(1)}_{MAX} \cdot c}{J_1} \qquad J_1 = \frac{\pi}{2} c^4 \qquad c = 100\ mm$$

$$\tau^{(2)}_{MAX} = \frac{T^{(2)}_{MAX} \cdot c}{J_2} \qquad J_2 = \frac{\pi}{2}(c^4 - b^4) \qquad b = 50\ mm$$

COMPARE THE TWO τ_{MAX}:

$$\tau_{MAX}^{①} = \tau_{MAX}^{②} = \tau_u$$

$$\frac{T_{MAX}^{①} \cdot c}{J_1} = \frac{T_{MAX}^{②} \cdot c}{J_2}$$

$$T_{MAX}^{②} = T_{MAX}^{①} \cdot \frac{J_2}{J_1} = T_{MAX}^{①} \cdot \frac{c^4 - b^4}{c^4} = T_{MAX}^{①}\left(1 - \frac{b^4}{c^4}\right)$$

$$= T_{MAX}^{①} \cdot 0.9375$$

$\Rightarrow$ reduction is 6.25%

Hooke's law for torsion

$$\phi = \int_0^L \frac{T}{GJ} dl \qquad \phi = \sum_i \frac{T_i L_i}{J_i G_i}$$

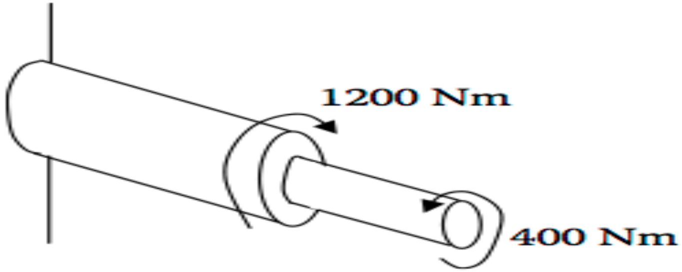
This form of Hooke's law can easily be used to measure the rigidity modulus of a material in a torsional test experiment by applying a torque onto the cylindrical test sample and measure the angle of twist. Since L and J are geometrical parameters that can be measure, G can be calculated from the slope of the experimental data.

For bars with varying cross-sections or materials properties we can sum up the twist angles as:

Normal load vs. Torsion

	Spring	Bar in Tension	Bar in Torsion
Geometric property	Δx	δ	ϕ
Materials property	$N.A.$	E	G
Hoooke's Law	$F = k\Delta x$	$P = \frac{EA}{L}\delta$	$T = \frac{GJ}{L}\phi$
Strain distribution	$N.A.$	$\tau \neq \tau(r)$	$\tau(r) = \frac{Tr}{J}$

Examples for Torsion: composite shaft

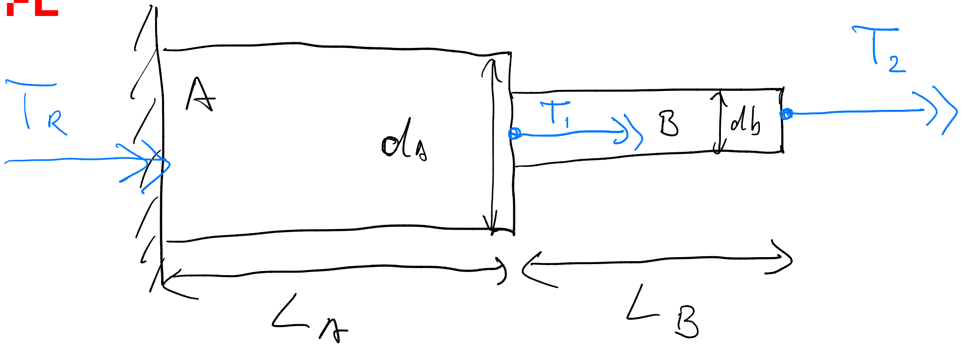


Two shafts (G = 28 GPa) *A* and *B* are joined and subjected to the torques shown in the figure below. Section *A* has a solid circular cross section with diameter 40 mm and is 160 mm long; *B* has a solid circular cross section with diameter 20 mm and is 120 mm long.

Find:

(a) the maximum shear stress in sections *A* and *B;* and

(b) the angle of twist of the right-most end of *B* relative to the wall.



GIVEN: GEOMETRY: $L_A = 160\ mm$

$L_B = 120\ mm$

$d_A = 40\ mm \Rightarrow c_A = 20\ mm$

$d_B = 20\ mm \Rightarrow c_B = 10\ mm$

LOADS: $T_1 = -1200\ Nm$ (right HAND RULE)

$T_2 = 400\ Nm$

MAT. PROP: $G = 28\ GPa$

ASKED: a) τ^A_{MAX}, τ^B_{MAX} b) ϕ AT END OF BAR

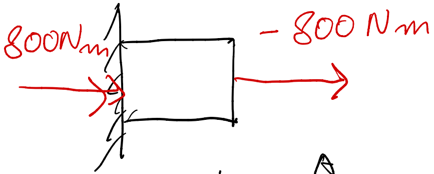
Gov. Princ: $\tau_{MAX} = \frac{Tc}{J}$ $\quad \phi = \frac{TL}{JG}$ $\quad J = \frac{\pi}{2} c^4$

Solution:

a) $\sum M = T_R + T_1 + T_2 = 0$

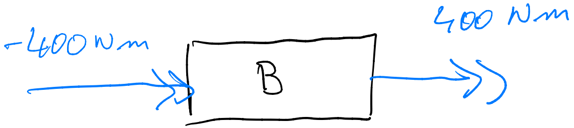
$$T_R = -(T_1 + T_2) = -(-1200 + 400) = +800\ Nm$$

METHOD OF SECTIONS



$$|\tau_{MAX}| = \frac{T \cdot c}{J} = \frac{T_R \cdot c}{\frac{\pi}{2} c^{3}} = \frac{2 T_R}{\pi c^3} = \frac{2 \cdot 800}{\pi (0{,}02)^3} = 63.7\ MPa$$

in element B:



$$\Rightarrow \tau_{MAX} = \frac{Tc}{J} = \frac{2T}{\pi c^3} = \frac{2\cdot 400}{\pi (0.01)^3} = 255\ MPa$$

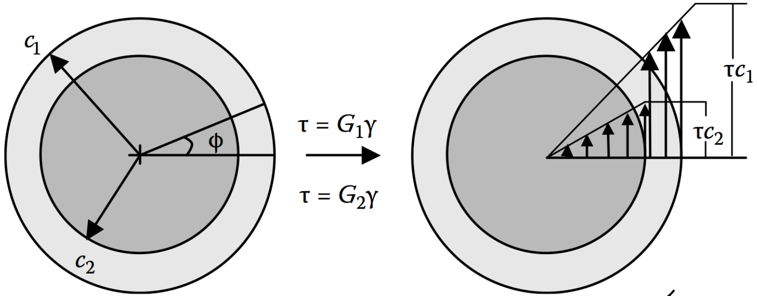
b) Superposition: $\phi = \phi_A + \phi_B$

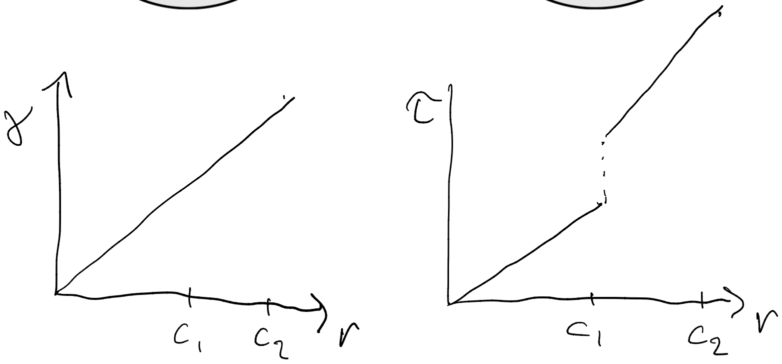
$$\phi_A = \frac{T_A \cdot L_A}{J_A\, G} = \frac{-800\ Nm \cdot 0.16\ m}{\frac{\pi}{2}(0{,}02)^4 \cdot 28\cdot 10^9\ Pa} = -1{,}82\cdot 10^{-2}\ rad$$

$$\phi_B = \frac{T_B \cdot L_B}{J_B\, G} = \frac{400\ Nm \cdot 0{,}12\ m}{\frac{\pi}{2}(0{,}01)^4 \cdot 28\cdot 10^9\ Pa} = +1.09\cdot 10^{-1}\ rad$$

$$\phi_{TOT} = \phi_A + \phi_B = 0{,}091\ rad = 5{,}21^\circ \text{ of twist CCW.}$$

Hooke's law for torsion- Multi material bars





If the bar consists of a core-shell structure, the strain is still linearly increasing.

But because of the change in materials properties, the stress is discontinuous.